

Shuffle Invariance of the Super-RSK Algorithm

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Abstract As in the (k, l) -RSK (Robinson-Schensted-Knuth) of [1], other super-RSK algorithms can be applied to sequences of variables from the set $\{t_1, \dots, t_k, u_1, \dots, u_l\}$, where $t_1 < \dots < t_k$, and $u_1 < \dots < u_l$. While the (k, l) -RSK of [1] is the case where $t_i < u_j$ for all i and j , these other super-RSK's correspond to all the $\binom{k+l}{k}$ shuffles of the t 's and u 's satisfying the above restrictions that $t_1 < \dots < t_k$ and $u_1 < \dots < u_l$. We show that the shape of the tableaux produced by any such super-RSK is independent of the particular shuffle of the t 's and u 's.

1 Introduction

We follow the tableaux-terminology of [7]. The classical Frobenius-Schur-Weyl theory shows how the SSYT (Semi-Standard-Young-Tableaux) determine the representations of $GL(m, \mathbb{C})$ (or $gl(m, \mathbb{C})$). Here $GL(m, \mathbb{C})$ ($gl(m, \mathbb{C})$) is the General Linear Lie group (algebra). Also, SYT (Standard-Young-Tableaux) play an important role here. The notion of (k, l) SSYT is introduced in [1], where similar relationships between such tableaux and the representations of $pl(k, l)$ are shown. Here $pl(k, l)$ is the General Linear Lie super-algebra.

The (k, l) SSYT are defined, via a (k, l) -RSK algorithm, as follows [1]. Fix integers $k, l \geq 0$, $k + l > 0$, and $k + l$ symbols $t_1, \dots, t_k, u_1, \dots, u_l$ such that $t_1 < \dots < t_k < u_1 < \dots < u_l$. Let

$$a_{k,l}(n) = \left\{ \left(\begin{array}{c} 1 \dots n \\ v_1 \dots v_n \end{array} \right) \middle| v_i \in \{t_1, \dots, t_k, u_1, \dots, u_l\} \right\}.$$

To map $a_{k,l}(n)$ to pairs of tableaux (P, Q) , apply to each $v \in a_{k,l}(n)$ the (k, l) -RSK, in which the usual RSK insertion algorithm [7] is applied to the t_i 's, and the conjugate correspondence (see [1]) is applied to the u_j 's; see the examples below. By the definitions of [1], the *insertion tableau*, $P = P(v)$, mapped from $v \in a_{k,l}(n)$, is (k, l) semistandard; that is, it satisfies the following three properties:

- (a) The “ t part” (i.e., the cells filled with t_i 's) is a tableau.
- (b) The t_i 's are nondecreasing in rows, strictly increasing in columns.
- (c) The u_j 's are nondecreasing in columns, strictly increasing in rows.

As in the usual correspondence, the *recording tableau*, $Q = Q(v)$, indicates the order in which the new cells were added to P . Clearly, Q is SYT having the same shape as that of P .

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A total order of $\{t_1, \dots, t_k, u_1, \dots, u_l\}$ which is compatible with $t_1 < \dots < t_k$ and $u_1 < \dots < u_l$, is called a *shuffle* (of t_1, \dots, t_k and u_1, \dots, u_l). For example, $t_1 < u_1 < u_2 < t_2$ is such a shuffle, compatible with $t_1 < t_2$ and $u_1 < u_2$. Clearly, there are $\binom{k+l}{k}$ such shuffles; of these, Berele and Regev chose to work with $t_1 < \dots < t_k < u_1 < \dots < u_l$, which we call *the (k, l) shuffle* (see [1, 2.4]). The shuffle $t_1 < u_1 < t_2 < u_2 < \dots < t_k < u_k$, with its corresponding SSYT, appears in section 4 of [3].

Let $I = I(k, l)$ denote the set of all such $\binom{k+l}{k}$ shuffles. Given $A \in I$, there is a corresponding A -RSK insertion algorithm; if $v \in a_{k,l}(n)$, then $v \xrightarrow[A]{} (P, Q)$ by that algorithm. $P = P_A = P(v, A)$ is the insertion tableau, and $Q = Q_A = Q(v, A)$ is the recording tableau. Here P is an A -SSYT; that is, it satisfies the following three properties.

- (a) P is weakly A -increasing in both rows and columns.
- (b) The t_i 's are strictly increasing in columns.
- (c) The u_j 's are strictly increasing in rows.

Example. Let $k = l = 2$, $A, B \in I = I(2, 2)$, where

$$A : t_1 < t_2 < u_1 < u_2 \quad \text{and} \quad B : u_1 < u_2 < t_1 < t_2.$$

Let

$$v = \begin{pmatrix} 1 \cdots \cdots 4 \\ u_2, t_1, t_2, u_1 \end{pmatrix}.$$

Then

$$v \xrightarrow[A]{} \boxed{u_2} \quad \boxed{t_1 \mid u_2} \quad \boxed{t_1 \mid t_2 \mid u_2} \quad \begin{array}{|c|c|c|} \hline t_1 & t_2 & u_2 \\ \hline u_1 & & \\ \hline \end{array} = P_A, \quad \text{while}$$

$$v \xrightarrow[B]{} \boxed{u_2} \quad \boxed{u_2 \mid t_1} \quad \boxed{u_2 \mid t_1 \mid t_2} \quad \begin{array}{|c|c|c|} \hline u_1 & u_2 & t_2 \\ \hline t_1 & & \\ \hline \end{array} = P_B$$

Thus $v \xrightarrow[A]{} (P_A, Q)$ and $v \xrightarrow[B]{} (P_B, Q)$, where

$$Q = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array}, \quad \text{and } P_A \text{ and } P_B \text{ are as above.} \quad \blacksquare$$

Definition. Denote by $\text{sh}(v, A) = \text{sh}(P_A)$ the shape of the insertion tableau $P(v, A) = P_A$ of $v \in a_{k,l}(n)$ under the A -RSK.

Given a shuffle $A \in I$ and the pair (P, Q) , where P is A -SSYT, Q is SYT, and $\text{sh}(P) = \text{sh}(Q)$, the A insertion algorithm can obviously be reversed. By standard arguments (see for example [7, chap. 7]) this yields

Theorem 1. *Let $A \in I$ be a shuffle. Then the A -RSK insertion algorithm $v \xrightarrow[A]{} (P_A, Q_A)$ is a bijection between $a_{k,l}(n)$ and*

$$\{(P_A, Q_A) \mid P_A \text{ is } A\text{-SSYT, } Q_A \text{ is SYT, } \text{sh}(P_A) = \text{sh}(Q_A)\}.$$

Remark. Denote such a tableau $P = (P_{i,j})$ and denote $<_A$ by $<$. Clearly, if $P_{i,j} = t_r$ then $P_{i,j-1} \leq P_{i,j} \leq P_{i,j+1}$ and $P_{i-1,j} < P_{i,j} < P_{i,j+1}$. Similarly, if $P_{i,j} = u_r$ then $P_{i,j-1} < P_{i,j} < P_{i,j+1}$ and $P_{i-1,j} \leq P_{i,j} \leq P_{i,j+1}$.

Denote by $\text{sh}(v, A)$ the shape of tableaux $P(v, A)$ and $Q(v, A)$. This brings us to our main result.

Theorem 2. *Let $v \in a_{k,l}(n)$, $A, B \in I$, $v \xrightarrow[A]{} (P_A, Q_A)$ and $v \xrightarrow[B]{} (P_B, Q_B)$. Then $\text{sh}(P_A) = \text{sh}(P_B)$. Consequently, $Q_A = Q_B$.*

In other words, the shape of the tableau obtained through any of the (k, l) -shuffle-RSK algorithms, is independent of the particular shuffle of the t 's and u 's.

Definition. Let $A \in I$ and $\lambda \vdash n$, i.e. a partition of n . Let $\mathfrak{S}_A(\lambda)$ denote the set of the A -SSYT of shape λ :

$$\mathfrak{S}_A(\lambda) = \{T \mid T \text{ is } A\text{-SSYT, } \text{sh}(T) = \lambda\}.$$

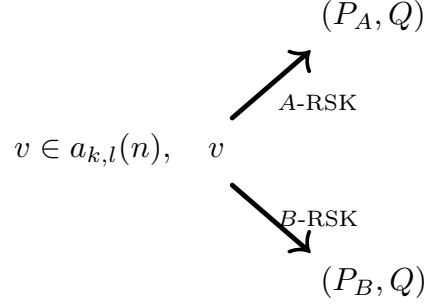
Recall the definition of $\text{type}(T)$ from [7, page 309].

Theorem 2 implies

Theorem 3 [6]. *Let $A, B \in I$, $\lambda \vdash n$. Then there exists a bijection $\varphi : \mathfrak{S}_A(\lambda) \rightarrow \mathfrak{S}_B(\lambda)$ such that for all $T \in \mathfrak{S}_A(\lambda)$, $\text{type}(T) = \text{type}(\varphi(T))$. (In fact, there exist (at least) d_λ such canonical bijections, where d_λ is the number of SYT's of shape λ .)*

Theorem 3 appears in [6], where it is proven by a different method. Our proof of the theorem is as follows.

Proof of Theorem 3. Is based on the following diagram:



Thus choose a SYT Q of shape λ . Given $P = P_A \in \mathfrak{S}_A(\lambda)$, get

$$(P_A, Q) \xrightarrow[\text{A-RSK}]{\text{inverse}} v \xrightarrow[\text{B-RSK}]{} (P_B, Q) .$$

This defines the bijection $\varphi = \varphi_Q : \varphi(P_A) = P_B$. Clearly, $\text{type}(P_A) = \text{type}(P_B)$ and by Theorem 2, $\text{sh}(P_A) = \text{sh}(P_B)$. ■

Recall from [2] the notation $w(T)$ for the weight of a tableau T . For example, let

$$T = \begin{array}{|c|c|c|c|} \hline t_1 & t_1 & u_2 & u_3 \\ \hline t_2 & t_3 & u_2 & \\ \hline u_1 & u_3 & & \\ \hline u_1 & & & \\ \hline \end{array}$$

then $w(T) = x_1^2 x_2 x_3 y_1^2 y_2^2 y_3^2$. Also recall the “hook” (or the “super”) Schur function

$$HS_\lambda(x; y) = HS_\lambda(x_1, \dots, x_k; y_1, \dots, y_l) \quad [1], [2].$$

When A is the shuffle $A_0 : t_1 < \dots < t_k < u_1 < \dots < u_l$, $HS_\lambda(x; y)$ is given by

$$HS_\lambda(x_1, \dots, x_k; y_1, \dots, y_l) = \sum_{T \in \mathfrak{S}_{A_0}(\lambda)} w(T)$$

[1, Thm. 6.10]. See also [4], [5] and [6].

It clearly follows from Theorem 3 that

Corollary 4. *For any $A \in I$,*

$$HS_\lambda(x_1, \dots, x_k; y_1, \dots, y_l) = \sum_{T \in \mathfrak{S}_A(\lambda)} w(T) .$$

Given a shuffle $A \in I$, the A -RSK is based on A , on the regular RSK for the t_i 's and the conjugate-regular RSK for the u_j 's.

In addition to the regular RSK, there is also the dual RSK [7, page 331]. Given the shuffle $A \in I$, this leads to four possible A -insertion algorithms: either the regular or the dual for the t_i 's, and either the conjugate regular or the conjugate dual for the u_j 's. In fact, the previous A -RSK is: (t -regular, u -conjugate-regular), which we denote as the (regular, regular)- A -RSK. Similarly, (t -regular, u -dual-conjugate) is the (regular, dual)- A -RSK. Similarly for the algorithms (dual, regular)- A -RSK and (dual, dual)- A -RSK. Each of these three new insertion algorithms exhibits a similar shape invariance under all shuffles $A \in I$.

Theorem 5.

(a) *Let $v \in a_{k,l}(n)$, and $A, B \in I$ such that*

$$v \xrightarrow{\text{(regular, regular)-}A\text{-RSK}} (P_A^*, Q_A^*), \quad v \xrightarrow{\text{(regular, regular)-}B\text{-RSK}} (P_B^*, Q_B^*).$$

Then $\text{sh}(P_A^) = \text{sh}(P_B^*)$. Consequently, $Q_A^* = Q_B^*$.*

(b) *Let $v \in a_{k,l}(n)$, and $A, B \in I$ such that*

$$v \xrightarrow{\text{(regular, dual)-}A\text{-RSK}} (P_A^*, Q_A^*), \quad v \xrightarrow{\text{(regular, dual)-}B\text{-RSK}} (P_B^*, Q_B^*).$$

Then $\text{sh}(P_A^) = \text{sh}(P_B^*)$. Consequently, $Q_A^* = Q_B^*$.*

(c) *Let $v \in a_{k,l}(n)$, and $A, B \in I$ such that*

$$v \xrightarrow{\text{(dual, regular)-}A\text{-RSK}} (P_A^*, Q_A^*), \quad v \xrightarrow{\text{(dual, regular)-}B\text{-RSK}} (P_B^*, Q_B^*).$$

Then $\text{sh}(P_A^) = \text{sh}(P_B^*)$. Consequently, $Q_A^* = Q_B^*$.*

(d) *Let $v \in a_{k,l}(n)$, and $A, B \in I$ such that*

$$v \xrightarrow{\text{(dual, dual)-}A\text{-RSK}} (P_A^*, Q_A^*), \quad v \xrightarrow{\text{(dual, dual)-}B\text{-RSK}} (P_B^*, Q_B^*).$$

Then $\text{sh}(P_A^) = \text{sh}(P_B^*)$. Consequently, $Q_A^* = Q_B^*$.*

Clearly, Theorem 5(a) is Theorem 2 above. The proof of Theorem 2 is given in the next section, which is the main body of this paper. First we describe the A -RSK algorithm in details. The main step in the proof of Theorem 2 is Lemma 2.15. It shows that a transposition of the variables in the shuffle (i.e a single change in the order of some t_i and u_j), does not alter the shape of the resulting tableaux. In section 3 we prove the remaining parts (b), (c), and (d) of Theorem 5, essentially by deducing them from Theorem 2.

2 Invariance of Shape

As in the (k, l) -RSK, the A -RSK insertion algorithm involves applying the usual RSK correspondence to the t_i 's, and the conjugate correspondence to the u_j 's. This is illustrated in the following example.

Definition 2.1. For $i, j \in \mathbb{Z}^+$, let $c(i, j)$ denote the cell in row i and column j of a given tableau.

Example 2.2. Under the shuffle $A = t_1 < u_1 < t_2 < u_2 < t_3$, perform the insertion

u_1	t_2	t_2
u_1	u_2	
t_3		

 $\leftarrow t_1.$

(a) $t_1 < u_1 \implies t_1$ occupies $c(1, 1)$. Now, a u_i is always bumped to the next column, hence u_1 is bumped to column 2.

(b) $u_1 < t_2 \implies u_1$ occupies $c(1, 2)$. Now, a t_i is always bumped to the next row, hence t_2 is bumped to row 2.

(c) $u_1 < t_2 < u_2 \implies t_2$ occupies $c(2, 2)$, bumping u_2 to column 3.

(d) $u_2 > t_2 \implies u_2$ settles in $c(2, 3)$.

(a)

t_1	t_2	t_2
u_1	u_2	
t_3		

(b)

t_1	u_1	t_2
u_1	u_2	
t_3		

(c)

t_1	t_2	t_2
u_1	t_2	
t_3		

(d)

t_1	u_1	t_2
u_1	t_2	u_2
t_3		

■

The proof of Theorem 2 will follow from the following analysis of the A -RSK algorithm.

Lemma 2.3. *Let p be an A -SSYT, $v \in \{t_1, \dots, t_k, u_1, \dots, u_l\}$. The insertion $P \leftarrow v$ is made of a sequence of several steps. In an intermediate m -th such a step, we have an A -SSYT \tilde{P} together with an element $P_{i,j}$ that was bumped from $c(i, j)$ by $\tilde{P}_{i,j}$, $\tilde{P}_{i,j} \leq_A P_{i,j}$, and we need to do the following insertion:*

- (a) *If $P_{i,j} = t_r$, insert it into the $i + 1$ -th row of \tilde{P} .*
- (b) *If $P_{i,j} = u_s$, insert it into the $j + 1$ -th column of \tilde{P} .*

We show that in both cases, the result would be an A -SSYT P^ , and – except for the last step – together with a new element $\tilde{P}_{i',j'}$ (bumped from $c(i', j')$), which is to be inserted into P^* . Moreover,*

- (1) *If $P_{i,j} = t_r$ then $c(i', j') = c(i + 1, j')$ and $j' \leq j$.*
- (2) *If $P_{i,j} = u_s$ then $c(i', j') = c(i', j + 1)$ and $i' \leq i$.*

then

Proof. Note that (2) is obtained from (1) by conjugation, hence it suffices to just prove (1).

Proof of (1): Denote the i -th row of \tilde{P} by

$$a_1 \cdots a_{j-1} \tilde{P}_{i,j} a_{j+1} \cdots a_g,$$

so $a_j = P_{i,j}$ and by assumption, $P_{i,j} = t_r$. Thus

$$\begin{array}{rcl} & & \vdots \\ & & a_1 \cdots a_{j-1} \tilde{P}_{i,j} a_{j+1} \cdots a_g \\ \tilde{P} & = & b_1 \cdots b_f \\ & & c_1 \cdots c_h \\ & & \vdots \end{array}$$

and $P_{i,j} = t_r$ is inserted into the $i + 1$ -th row $b_1 \cdots b_f$.

Let $b_{j'-1} \leq P_{i,j} < b_{j'}$, so in P^* , the $i + 1$ -th row is

$$b_1 \cdots b_{j'-1} P_{i,j} b_{j'+1} \cdots b_f.$$

Since $\tilde{P}_{i,j}$ bumped $P_{i,j}$, we have $\tilde{P}_{i,j} < P_{i,j}$. Since $a_j = P_{i,j} = t_r$, hence $P_{i,j} < b_j$. Together with $b_{j'-1} \leq P_{i,j} < b_{j'}$, this implies that $j' \leq j$, hence

$$\begin{array}{rcl} & & \vdots \\ & & a_1 \cdots a_{j'-1} a_{j'} a_{j'+1} \cdots \tilde{P}_{i,j} a_{j+1} \cdots a_g \\ P^* & = & b_1 \cdots b_{j'-1} P_{i,j} b_{j'+1} \cdots b_j b_{j+1} \cdots b_f \\ & & c_1 \cdots c_{j'-1} c_{j'} c_{j'+1} \cdots c_j c_{j+1} \cdots c_h \\ & & \vdots \end{array}$$

By the induction assumption on \tilde{P} , we only need to verify that the part

$$\begin{array}{c} a_{j'} \\ P_{i,j} \\ c_{j'} \end{array}$$

of the j' -th column is A -semistandard, i.e.: since $P_{i,j} = t_r$, we need to show that $a_{j'} \leq \tilde{P}_{i,j} < c_{j'}$. This follows from $a_{j'} \leq \tilde{P}_{i,j} < P_{i,j} = t_r < b_{j'} \leq c_{j'}$. ■

Definition 2.4. Two shuffles $A, B \in I$ are *adjacent* if there exist t_i and u_j such that

- 1) $t_i < u_j$ in A .
- 2) $u_j < t_i$ in B .
- 3) All other pairs have the same order relations in A and in B .

In that case, call A and B (t_i, u_j) -adjacent. Thus A and B differ by the transposition (t_i, u_j) .

Remark 2.5. Trivially, for any $A, B \in I$ there exist $A_0, A_1, \dots, A_n \in I$ such that $A_0 = A$, $A_n = B$, and A_r is adjacent to A_{r+1} , $0 \leq r \leq n-1$. Thus to prove Theorem 1, it suffices to show that for all $v \in a_{k,l}(n)$ and for every pair (A, B) of adjacent shuffles, $\text{sh}(v, A) = \text{sh}(v, B)$. Therefore for the rest of this section, let $A, B \in I$ be (t_i, u_j) -adjacent, with $t_i <_A u_j$ and $u_j <_B t_i$.

Lemma 2.6. Let $A \in I$, $w \in a_{k,l}(n)$, and for some $x \in \{t_1, \dots, t_k, u_1, \dots, u_l\}$, let w' be the sequence obtained by omitting from w all elements A -greater than x . Let P_A and P'_A be the insertion tableaux obtained from w and w' respectively under shuffle A . Then P'_A is a subtableau of P_A .

Proof. Let $w \xrightarrow{\text{A-RSK}} P_A$; $P : \emptyset, P_1, P_2, \dots, P_n = P_A$, and similarly let $w' \xrightarrow{\text{A-RSK}} P'_A$; $P' : \emptyset, P'_1, P'_2, \dots, P'_m = P'_A$ ($m = |w'|$).

Assume P'_i is a subtableau of P_{j_i} , and insert (a corresponding) y in w .

If $x <_A y$, y is not in w' so P'_i is not affected. Also, inserting y into P_{j_i} , y does not affect the subtableau $P'_i \subseteq P_{j_i}$, since y bumps only elements that are A -greater than itself.

A similar argument applies when $y \leq x$: now y is also in w' , and is inserted into P'_i and into P_{j_i} . Clearly, in P_{j_i} it is also inserted into the subtableau $P'_i \subseteq P_{j_i}$, and the proof follows. ■

Corollary 2.7. *Let $A, B \in I$ be (t_i, u_j) -adjacent, $v \in a_{k,l}(n)$, $v \xrightarrow[A]{\longrightarrow} (P_A, Q_A)$ and $v \xrightarrow[B]{\longrightarrow} (P_B, Q_B)$. Then the elements that are both A -less and B -less than t_i and u_j form identical subtableaux in P_A and P_B .*

Proof. Denote by v' the sequence obtained by omitting from v all elements (A - and B -) greater than or equal to t_i and u_j . By (t_i, u_j) -adjacency, the largest element smaller than t_i and u_j , in both A and B , is the same element x . Moreover, v' is obtained by omitting from v all elements which are (A - or B -) greater than x . Let P'_A and P'_B denote the insertion tableaux of v' under shuffles A and B respectively. Then by Lemma 2.6, P'_A and P'_B are subtableaux of P_A and P_B respectively. But the elements that are A - or B -less than t_i and u_j are ordered identically in A and B , so $P'_A = P'_B$. ■

Notation. As above, let $A, B \in I$ be two shuffles that are (t_i, u_j) -adjacent: $t_i < u_j$ in A and $u_j < t_i$ in B . Let $v \in a_{k,l}(n)$, and denote $v \xrightarrow[A]{\longrightarrow} (P_A, Q_A)$ and $v \xrightarrow[B]{\longrightarrow} (P_B, Q_B)$.

Notation. Given the tableau P_A (and similarly for P_B), let regions 1, 2 and 3 denote, respectively, the regions occupied (1) by elements less than t_i and u_j , (2) by t_i and u_j , and (3) by elements greater than t_i and u_j .

Example 2.8. Let $v = u_1 t_3 t_2 u_2 t_2 u_1 t_1$, and let

$$\begin{aligned} A &= t_1 < u_1 < t_2 < u_2 < t_3 \\ B &= t_1 < u_1 < u_2 < t_2 < t_3. \end{aligned}$$

Then A and B are (t_i, u_j) -adjacent, with $t_i = t_2$ and $u_j = u_2$, and

$$P_A = \begin{array}{|c|c|c|} \hline t_1 & u_1 & t_2 \\ \hline u_1 & t_2 & u_2 \\ \hline t_3 & & \\ \hline \end{array}, \quad P_B = \begin{array}{|c|c|c|} \hline t_1 & u_1 & u_2 \\ \hline u_1 & t_2 & t_2 \\ \hline t_3 & & \\ \hline \end{array}.$$

In both tableaux, region 1 contains the elements t_1 and u_1 , region 2 contains t_2 and u_2 , and region 3 contains t_3 . Note that in this example, regions 1 and 3 are the same in P_A as in P_B , and region 2 is identically shaped in P_A and P_B . We shall show that this is always true. ■

By Lemma 2.6, both region 1, as well as the union of regions 1 and 2, form subtableaux in P . It is easy to check that region 2 does not contain the configuration

$$\begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array}.$$

If it does, assume $d = t_i$. Then $b = u_j$, so $u_j < t_i$, and $a \neq t_i, u_j$. Similarly if $d = u_j$. It follows that region 2 forms part of the rim of the subtableaux which is the union of regions 1 and 2.

Remark 2.9. Note that (part of) region 2 in P_A (i.e. $t_i < u_j$) always looks like

$$\begin{array}{c} t_i \cdots \cdots t_i \\ u_j \\ \vdots \\ t_i \cdots \cdots t_i u_j \\ u_j \\ \vdots \\ u_j \end{array}$$

Namely: Except possibly for the rightmost element, all other elements in a row are t_i 's. Similarly, except for possibly the top element, all other elements in a column are u_j 's.

Similarly, in P_B (i.e. $u_j < t_i$), part of region 2 looks like

$$\begin{array}{c} u_j t_i \cdots \cdots t_i \\ \vdots \\ u_j t_i \cdots \cdots t_i \\ \vdots \\ u_j \end{array}$$

Denote $v = v_1 \cdots v_n$. The tableau P_A is created by applying the A -RSK insertion algorithm to each of v_1, \dots, v_n successively. For each v_m , let $l_{m(A)}$ denote the length of the insertion path [7, page 317] of v_m under shuffle A – that is, the number of insertion steps that occur when v_m is inserted while forming P_A . The total number of insertion steps involved in the formation of P_A is thus $s_A = \sum_{m=1}^n l_{m(A)}$. For every $r \in \{1, \dots, s_A\}$, let P_A^r be the insertion tableau as it appears immediately after insertion step r .

Similarly, under shuffle B , the length of the insertion path of v_m into P_B is $l_{m(B)}$, and the total number of insertion steps involved in forming P_B is $s_B = \sum_{m=1}^n l_{m(B)}$, with P_B^r denoting the insertion tableau after insertion step r .

Example 2.10. As in Example 2.8, let $v = v_1 \cdots v_7 = u_1 t_3 t_2 u_2 t_2 u_1 t_1$, and let $A = t_1 < u_1 < t_2 < u_2 < t_3$. Then tableau P_A is formed by the A -RSK as follows (ignore the underlines).

<u>u_1</u>	u_1	<u>t_3</u>	u_1	<u>t_2</u>	u_1	t_2	
				<u>t_3</u>	<u>u_2</u>		
					<u>t_3</u>		
u_1	t_2	<u>t_2</u>	u_1	t_2	<u>t_1</u>	<u>u_1</u>	t_2
u_2			<u>u_1</u>	<u>u_2</u>	u_1	<u>t_2</u>	<u>u_2</u>
t_3			t_3		t_3		

For all $i \in \{1, \dots, 7\}$, the underlined elements in tableau i lie in the insertion path of element v_i . Thus $l_{1(A)} = l_{2(A)} = l_{5(A)} = 1$, $l_{3(A)} = l_{4(A)} = l_{6(A)} = 2$, $l_{7(A)} = 4$, and $s_A = \sum_{i=1}^7 l_{i(A)} = 13$. If for example, $r = 7 = \sum_{i=1}^5 l_{i(A)}$, then we have

$$P^r = \begin{array}{|c|c|c|} \hline u_1 & t_2 & t_2 \\ \hline u_2 & & \\ \hline t_3 & & \\ \hline \end{array}, \quad P^{r+1} = \begin{array}{|c|c|c|} \hline u_1 & t_2 & t_2 \\ \hline u_1 & & \\ \hline t_3 & & \\ \hline \end{array} \quad \blacksquare$$

Example 2.11. Let $k = l = 1$, $A : t < u$, $B : u < t$, $v = v_1 v_2 = tu$. Then

$$P_A : \emptyset, \begin{array}{|c|} \hline \underline{t} \\ \hline \end{array}, \begin{array}{|c|} \hline t \\ \hline \underline{u} \\ \hline \end{array}; \quad l_{1(A)} = l_{2(A)} = 1,$$

$$P_B : \emptyset, \begin{array}{|c|} \hline \underline{t} \\ \hline \end{array}, \begin{array}{|c|} \hline \underline{u} \\ \hline \underline{t} \\ \hline \end{array}; \quad l_{1(B)} = 1, l_{2(B)} = 2. \quad \blacksquare$$

Definition 2.12. For $p, q \in \mathbb{Z}^+$, we say that $P_A^p \sim P_B^q$ (with respect to the formations of P_A and P_B) if:

1. Regions 1 and 3 are identical in P_A^p and P_B^q .
2. Region 2 is identically shaped in P_A^p and P_B^q ; moreover, in each connected component of that region 2, the number of t_i 's (hence of u_j 's) in P_A^p equals the number of t_i 's (hence of u_j 's) in P_B^q .
3. Either $p = s_A$ and $q = s_B$, or both $p < s_A$ and $q < s_B$. In the latter case, the next insertion step involves inserting the same element into the same row (or column) in both tableaux.

Example 2.13. The tableaux of Example 2.8 satisfy $P_A \sim P_B$. Regions 1 and 3 in the two tableaux are identical, satisfying property 1 of Definition 2.12. Region 2 consists of one component which is identically shaped, and contains exactly one t_i and one u_j , in both tableaux. This verifies property 2. Since both tableaux correspond to $p = s_A$ and $q = s_B$, property 3 is satisfied as well. \blacksquare

Lemma 2.14. *For any shuffle $A \in I$, and for all $p \in \{2, \dots, s_A\}$ and $r, s \in \mathbb{Z}^+$, if $c(r, s)$ contains some w in P_A^{p-1} , then $c(r, s)$ contains some $z \leq_A w$ in P_A^p .*

Conversely, if $c(r, s)$ contains some element z in P_A^p , then $c(r, s)$ was either empty or contained some $w \geq_A z$ in P_A^{p-1} .

Proof. Follows from the A -RSK algorithm. ■

The Proof of Theorem 2 clearly follows from

Lemma 2.15. *Let $A, B \in I$ be (t_i, u_j) -adjacent, $v \in a_{k,l}(n)$, $v \xrightarrow{\text{A-RSK}} (P_A, Q_A)$ and $v \xrightarrow{\text{B-RSK}} (P_B, Q_B)$, then $P_A \sim P_B$.*

Proof. We prove that $P_A \sim P_B$, by induction on the insertion steps of P_A and P_B . Trivially $P_A^1 = P_B^1$. Now let $p \in \{1, \dots, s_A - 1\}$, $q \in \{1, \dots, s_B - 1\}$ and assume that 1) $P_A^p \sim P_B^q$, and also 2) either $P_A^{p-1} \sim P_B^{q-1}$ or $P_A^{p-1} \sim P_B^{q-2}$ or $P_A^{p-2} \sim P_B^{q-1}$. We show that this implies that either $P_A^{p+1} \sim P_B^{q+1}$ or $P_A^{p+1} \sim P_B^{q+2}$ or $P_A^{p+2} \sim P_B^{q+1}$. This clearly implies the proof of the lemma (by induction on $p + q$).

Note that if $P_A^p \sim P_B^q$, then by 2.12.3, step $p + 1$ in P_A and step $q + 1$ in P_B are identical; that is, the same element, x , is inserted into the same row (or column) in both tableaux. We assume that that x is a t -element, and therefore enters some row, denoted *row* r ; the case where x is a u -element is analogous. Since $P_A^p \sim P_B^q$, row r is empty in P_A^p if and only if it is empty in P_B^q . The case where row r is empty is trivial, so we assume throughout that row r is nonempty in P_A^p and P_B^q .

1. Suppose that under both shuffles A and B , $x > t_i$ and u_j . Since $P_A^p \sim P_B^q$, the last nonempty cell in row r must be in the same region in both P_A^p and P_B^q , and if it is in region 3, then it must be occupied by the same element in both tableaux.

Case 1.1: Row r in P_A^p (and in P_B^q) terminates with an element less than or equal to x . In this case, x is affixed to the end of the row in both tableaux, so P_A^{p+1} and P_B^{q+1} have the same shape and clearly satisfy properties 1 and 2 of Definition 2.12. Let m denote the size of P_A^{p+1} and P_B^{q+1} . If $m = n$, which is the size of P_A and P_B , then the insertion algorithm terminates here. Otherwise, the next step is to begin v_{m+1} 's insertion path by inserting v_{m+1} into either the first row or the first column in both tableaux. This verifies 2.12.3 and we have $P_A^{p+1} \sim P_B^{q+1}$.

Case 1.2: Row r in P_A^p contains an element $z > x$ (under both A and B). Since $P_A^p \sim P_B^q$, the same is true in P_B^q . In this case, x bumps an element greater than itself – a region-3 element – and occupies its cell in both tableaux. Thus both the cell occupied by x and the element bumped by x are identical in the two tableaux, which 2.12.3. Since 2.12.1 and 2.12.2 clearly hold, it follows that $P_A^{p+1} \sim P_B^{q+1}$.

2. Suppose that $x = t_i$. During step $P_B^q \rightarrow P_B^{q+1}$, $x = t_i >_B u_j$ bumps the first region-3 element in row r , or if no such element exists, x occupies the first empty cell in that row. Let $c(r, s)$ be the cell occupied by x in P_B^{q+1} .

Case 2.1: In row r of P_A^p , region 2 either terminates with t_i or does not appear at all in that row. Then x occupies $c(r, s)$ also in P_A^{p+1} (and bumps the same element as in P_B^{q+1}), so $P_A^{p+1} \sim P_B^{q+1}$.

Case 2.2: In P_A^p , the last region-2 element in row r is u_j . Let this u_j be in $c(r, s')$. Since $P_A^p \sim P_B^q$, $c(r, s')$ is the last region-2 cell in row r in both tableaux. Since in $P_B^q \rightarrow P_B^{q+1}$, x was inserted into $c(r, s)$, we have $s = s' + 1$. Thus u_j is in $c(r, s - 1)$ and is bumped by $x = t_i$ to column s during $P_A^p \rightarrow P_A^{p+1}$. We prove that in such a case, $P_A^{p+2} \sim P_B^{q+1}$. To do so, we show that

2.2.1: In $P_A^{p+1} \rightarrow P_A^{p+2}$, u_j settles in $c(r, s)$, to the immediate right of x .

2.2.2: This implies that 2.12.2 for $P_A^{p+2} \sim P_B^{q+1}$ is satisfied.

2.2.3: Both 2.12.1 and 2.12.3 for $P_A^{p+2} \sim P_B^{q+1}$ are satisfied.

Proof of 2.2.1: If $r = 1$, then u_j clearly settles in $c(r, s)$ in P_A^{p+2} . We therefore assume that $r > 1$.

To prove that u_j settles in $c(r, s)$ in P_A^{p+2} , we need only to show that $c(r - 1, s)$ in P_A^{p+1} contains an element $b \leq u_j$, since $c(r, s)$ in P_A^{p+1} contains some element $z >_A u_j$. Now, since $r > 1$, $x = t_i$ arrived at row r in P_A^p (and similarly in P_B^q) after being bumped from row $r - 1$ of P_A^{p-1} . Let $c(r - 1, h)$ be the cell occupied by x in P_A^{p-1} , before it was bumped from row $r - 1$.

$$P_A^{p-1} \xrightarrow[\text{from } c(r-1, h)]{x \text{ is bumped}} P_A^p \xrightarrow[\text{into } c(r, s-1)]{x \text{ is inserted}} P_A^{p+1} \xrightarrow[\text{into column } s]{u_j \text{ is inserted}} P_A^{p+2}$$

Since x is inserted into $c(r, s - 1)$ of P_A^{p+1} , Lemma 2.3 implies that $s - 1 \leq h$. If $s \leq h$, then $c(r - 1, s)$ was occupied by an element less than or equal to x in P_A^{p-1} , and this continues to be true in P_A^p and P_A^{p+1} , which implies our claim (that $b \leq u_j$). On the other hand, suppose that $h = s - 1$. In this case, we prove that $c(r - 1, s)$ contains an element less than or equal to u_j by showing that otherwise we would have a contradiction to the induction assumptions. Our proof of this is illustrated by the following figures, each of which consists of the block of cells in rows $r - 1$ and r and columns $s - 1$ and s in the corresponding tableaux.

$$P_A^{p-1} : \begin{array}{c} \vdots \\ s \\ \vdots \end{array} \begin{array}{|c|c|} \hline x & b \\ \hline \hline r & \cdots \end{array}, \quad P_A^p : \begin{array}{c} \vdots \\ s \\ \vdots \end{array} \begin{array}{|c|c|} \hline & b \\ \hline \hline r & \cdots \end{array} \begin{array}{|c|c|} \hline & u_j \\ \hline \end{array},$$

$$\begin{array}{c}
P_A^{p+1} : \quad \begin{array}{c} s \\ \vdots \\ \begin{array}{|c|c|} \hline & b \\ \hline \end{array} \\ r \cdots \begin{array}{|c|c|} \hline x & \\ \hline \end{array} \end{array} \\
\\
P_B^{q-2} : \quad \begin{array}{c} s \\ \vdots \\ \begin{array}{|c|c|} \hline & z \\ \hline \end{array} \\ r \cdots \begin{array}{|c|c|} \hline & \\ \hline \end{array} , \quad P_B^{q-1} : \quad \begin{array}{c} s \\ \vdots \\ \begin{array}{|c|c|} \hline & e \\ \hline \end{array} \\ r \cdots \begin{array}{|c|c|} \hline & \\ \hline \end{array} . \end{array}
\end{array}$$

So, assume that in P_A^{p-1} , x occupied $c(r-1, s-1)$, and $c(r-1, s)$ was either empty or contained an element $b > u_j$ (a region-3 element). We show that this contradicts the claim of the induction hypothesis, that either $P_A^{p-1} \sim P_B^{q-1}$, $P_A^{p-1} \sim P_B^{q-2}$, or $P_A^{p-2} \sim P_B^{q-1}$.

$$\begin{array}{ccccc}
P_B^{q-2} & \xrightarrow[e \text{ is inserted into } c(r-1, s)]{} & P_B^{q-1} & \xrightarrow[x \text{ is bumped from } c(r-1, s')]{ } & P_B^q & \xrightarrow[x \text{ is inserted into } c(r, s)]{} & P_B^{q+1}
\end{array}$$

Now in P_B^{q+1} , $x = t_i$ occupies $c(r, s)$, so $c(r-1, s)$ is occupied by an element less than x . Lemma 2.3 implies that in P_B^q , x occupied $c(r-1, s')$, where $s \leq s'$, and this implies that in P_B^{q-1} , $c(r-1, s)$ contained an element $e \leq x$, a region-1 or -2 element. But the same cell in P_A^{p-1} was either empty or contained a region-3 element, and by Lemma 2.14, the same must have been true in P_A^{p-2} . Thus the above assumption, that $h = s-1$, implies that neither $P_A^{p-1} \sim P_B^{q-1}$ nor $P_A^{p-2} \sim P_B^{q-1}$ is satisfied. We show now that it also implies that $P_A^{p-1} \sim P_B^{q-2}$.

Suppose that $P_A^{p-1} \sim P_B^{q-2}$ is satisfied. Denote by m the size of P_A^{p-1} and P_B^{q-2} . By assumption, in P_A^{p-1} – hence also in $P_B^{q-2} \sim P_A^{p-1}$ – x occupies $c(r-1, s-1)$ and $c(r-1, s)$ is either empty or contains a region-3 element $z > x$. But we saw above that in P_B^{q-1} , $c(r-1, s)$ contained $e \leq x$. It follows that insertion step $P_B^{q-2} \rightarrow P_B^{q-1}$ consisted of e being inserted into $c(r-1, s)$, and – if $c(r-1, s)$ were previously occupied – of bumping from it some $z > x$. If e bumped some z , then in $P_B^{q-1} \rightarrow P_B^q$, z would have been inserted into either row r or column $s+1$. But we saw earlier that x was bumped to row r in $P_B^{q-1} \rightarrow P_B^q$, which implies that z must have bumped x during this step. This leads to a contradiction, since z could not have bumped $x < z$. Thus when e occupied $c(r-1, s)$ during $P_B^{q-2} \rightarrow P_B^{q-1}$, it did not bump any element; the cell was previously empty. By occupying an empty cell, e increased the size of the tableau from m to $m+1$: $|P_B^{q-2}| = m$, $|P_B^{q-1}| = m+1$. Hence $|P_B^q| \geq m+1$.

On the other hand, P_A^{p-1} is of size m , and during $P_A^{p-1} \rightarrow P_A^p$, x was bumped from row $r-1$ by some smaller element, so the size of the tableau did not change. It follows that $|P_A^p| = m < |P_B^q|$, contradicting $P_A^p \sim P_B^q$.

It follows that $c(r-1, s)$ in P_A^{p+1} contains an element $b \leq u_j$, and this implies that when u_j enters column s in $P_A^{p+1} \rightarrow P_A^{p+2}$, it settles in $c(r, s)$, to the right of x . This completes the proof of 2.2.1.

Proof of 2.2.2: By 2.2.1, u_j settles in $c(r, s)$ in P_A^{p+2} . We show that this implies that 2.12.2 for $P_A^{p+2} \sim P_B^{q+1}$ is satisfied. In the diagrams below the proof, the cells outside of region 2 are marked with a \star .

Recall that by “Case 2.2”, u_j is located in $c(r, s-1)$ in P_A^p , so $c(r, s-1)$ is part of a connected component of region 2. Let τ = the number of t_i ’s and μ = the number of u_j ’s in this connected component. By 2.12.2 of $P_A^p \sim P_B^q$, region 2 of P_B^q contains a corresponding connected component (in the same cells as that of P_A^p), with τ t_i ’s and μ u_j ’s.

Since u_j is located in $c(r, s-1)$ in P_A^p , by strict row inequality, $c(r, s)$ is either empty or in region 3. Similarly, since $x = t_i$ occupies $c(r, s-1)$ in $P_A^p \rightarrow P_A^{p+1}$, $c(r-1, s-1)$ must contain a region-1 element in P_A^{p+1} (and in P_A^p).

$$P_A^p : \begin{array}{c} \vdots \\ \begin{array}{|c|c|} \hline \star & \\ \hline u_j & \star \\ \hline \end{array} \end{array}$$

Thus, if $c(r-1, s)$ is in region 2 in P_A^p and P_B^q , then in both tableaux it is part of a connected component distinct from that of $c(r, s-1)$. This other component consists of τ' t_i ’s and μ' u_j ’s. (If $c(r-1, s)$ is not in region 2, then we let $\tau' = \mu' = 0$.)

Since $c(r, s)$ is either empty or in region 3 in P_A^p (and thus in P_B^q), it follows that the same is true for $c(r, s+1)$ and $c(r+1, s)$. By the assumption at the beginning of case 2, during $P_B^q \rightarrow P_B^{q+1}$, $x = t_i$ bumps some region-3 element z from $c(r, s)$, thereby adding $c(r, s)$ to region 2. But $c(r, s)$ is adjacent to both $c(r, s-1)$ and $c(r-1, s)$, so when it joins region 2, it combines their respective connected components into a single larger one. Since neither $c(r, s+1)$ nor $c(r+1, s)$ is in region 2, it follows that in P_B^{q+1} , $c(r, s)$ becomes part of a connected component of region 2, consisting of $\tau + \tau' + 1$ t_i ’s and $\mu + \mu'$ u_j ’s.

$$P_B^{q+1} : \begin{array}{c} \vdots \\ \begin{array}{|c|c|c|} \hline \star & & \\ \hline & x & \star \\ \hline & \star & \\ \hline \end{array} \end{array}$$

Similarly, during $P_A^p \rightarrow P_A^{p+1} \rightarrow P_A^{p+2}$, $x = t_i$ bumps u_j from $c(r, s-1)$, and u_j bumps z from $c(r, s)$, so the only change in the shape of region 2 in P_A^{p+2} is the addition of $c(r, s)$. Thus $c(r, s)$ of P_A^{p+2} is part of a connected

component of region 2, also containing $\tau + \tau' + 1$ t_i 's and $\mu + \mu'$ u_j 's, and this component is identically shaped to the corresponding component of P_B^{q+1} , so 2.12.2 of $P_A^{p+2} \sim P_B^{q+1}$ is satisfied. This completes the proof of 2.2.2.

Proof of 2.2.3: Property 2.12.1 is satisfied for $P_A^{p+2} \sim P_B^{q+1}$, since in both $P_B^q \rightarrow P_B^{q+1}$ and $P_A^p \rightarrow P_A^{p+1} \rightarrow P_A^{p+2}$, region 1 is unchanged, and the only change in region 3 is the elimination of $c(r, s)$.

Since the same element z is bumped from $c(r, s)$ in $P_A^{p+1} \rightarrow P_A^{p+2}$ and $P_B^q \rightarrow P_B^{q+1}$, 2.12.3 is satisfied, and the proof of 2.2.3 is complete.

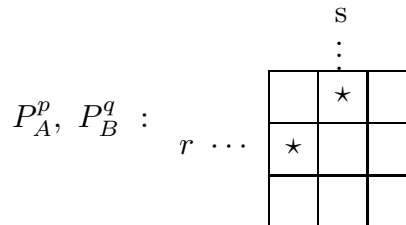
It follows that $P_A^{p+2} \sim P_B^{q+1}$.

3. Suppose that $x = t_a < t_i$.

Case 3.1: Row r in P_A^p terminates with $z \leq x$. Thus z is in region 1, so by 2.12.1 of $P_A^p \sim P_B^q$, row r in P_B^q terminates with z . In such a case, x is affixed to the end of row r in both tableaux, so P_A^{p+1} and P_B^{q+1} have the same shape, and clearly satisfy 2.12.1 and 2.12.2. Let m denote the size of P_A^{p+1} and P_B^{q+1} . If $m = n$, which is the size of P_A and P_B , then the insertion algorithm terminates here. Otherwise, the next step is to begin v_{m+1} 's insertion path by inserting v_{m+1} into either the first row or the first column in both tableaux. This verifies 2.12.3 and we have $P_A^{p+1} \sim P_B^{q+1}$.

Case 3.2: Row r in P_A^p (and hence in P_B^q) contains an element greater than x . In both tableaux, x bumps from row r the leftmost element greater than itself. By 2.12.1 and 2.12.2 of $P_A^p \sim P_B^q$, the same cell – denoted $c(r, s)$ – becomes occupied by x in both tableaux. Thus if the element bumped by x is identical in the two tableaux, then $P_A^{p+1} \sim P_B^{q+1}$.

Suppose, however, that x bumps different elements from the cell(s) $c(r, s)$ of P_A^p and P_B^q . By $P_A^p \sim P_B^q$, these must be t_i and u_j . Since $x = t_a$ occupies $c(r, s)$ in P_A^{p+1} , Lemma 2.3 implies that in P_A^{p-1} , x occupied $c(r-1, s')$ with $s \leq s'$. Thus the $c(r-1, s)$ element g in P_A^{p-1} was $g \leq x < u_j, t_i$, so $c(r-1, s)$ was a region-1 cell. Since x was subsequently bumped from row $r-1$ by an element smaller than itself, it follows that $c(r-1, s)$ is a region-1 cell also in P_A^p (and P_B^q). Similarly, since $x = t_a < t_i$ settles in $c(r, s)$ in $P_A^p \rightarrow P_A^{p+1}$, $c(r, s-1)$ is a region-1 cell in P_A^{p+1} (and P_B^{q+1}), and in P_A^p (and P_B^q).



(The stars represent region-1 elements.)

Now, since $c(r, s)$ is in region 2 in both P_A^p and P_B^q , by 2.12.2, it is part of a connected component of region 2 which is identically shaped, and contains the same number of t_i 's and u_j 's, in both tableaux. But $c(r, s)$ contains t_i in one tableau and u_j in the other, so it follows that at least one of $c(r, s+1)$ and $c(r+1, s)$ is in region 2 in P_A^p and P_B^q . By strict row and column inequality, this implies that $c(r, s)$ contains t_i in P_A^p and u_j in P_B^q .

Denote by C the connected component of region 2 containing $c(r, s)$. Consider the subcomponent C_1 , consisting of all cells in C which are to the right of or above $c(r, s)$. In P_A^p , let $\alpha_A = \#t_i$'s, and $\beta_A = \#u_j$'s in C_1 ; define α_B and β_B similarly in P_B^q . If $c(r, s+1)$ is not in region 2, then C_1 is empty and $\alpha_A = \beta_A = \alpha_B = \beta_B = 0$. On the other hand, if C_1 is nonempty, then by strict row and column inequality, every northwest proper corner cell of C_1 contains t_i in P_A^p and u_j in P_B^q .

$$P_A^p : \begin{array}{|c|c|} \hline t_i & \\ \hline & \\ \hline \end{array} \cdots, \quad P_B^q : \begin{array}{|c|c|} \hline u_j & \\ \hline & \\ \hline \end{array} \cdots$$

$$\vdots \qquad \qquad \qquad \vdots$$

Similarly, every southeast proper corner cell of C_1 contains u_j in P_A^p and t_i in P_B^q .

$$P_A^p : \begin{array}{ccc} & & \vdots \\ & & \square \\ \cdots & & \square \\ & & u_j \end{array}, \quad P_B^q : \begin{array}{ccc} & & \vdots \\ & & \square \\ \cdots & & \square \\ & & t_i \end{array}$$

Consider the top row of C_1 . If it contains more than one cell, then its leftmost cell is a northwest corner. Thus the structure of C_1 is as in the following diagram, where for example, a cell marked t_i/u_j contains t_i in P_A^p and u_j in P_B^q . (A mark of ? denotes that a cell may contain either t_i or u_j .) and elements

$$\begin{array}{c} t_i/u_j \cdots \cdots ?/t_i \\ \vdots \\ \cdots \cdots u_j/t_i \end{array}$$

??

On the other hand, if the top row of C_1 contains only one cell, then the structure of C_1 is

$$\begin{array}{c}
?/u_j \\
\vdots \\
t_i/u_j \cdots \cdots u_j/t_i \\
\vdots \\
\cdots \cdots u_j/t_i
\end{array}$$

In both cases it follows that $\alpha_B - \alpha_A = \beta_A - \beta_B \in \{0, 1\}$. We prove that

$$3.2.1: \alpha_B - \alpha_A = \beta_A - \beta_B = 1 \implies P_A^{p+1} \sim P_B^{q+2}.$$

$$3.2.2: \alpha_B - \alpha_A = \beta_A - \beta_B = 0 \implies P_A^{p+2} \sim P_B^{q+1}.$$

Let C_2 be the subcomponent of C consisting of all cells below or to the left of $c(r, s)$. Let $\gamma_A = \#t_i$'s and $\delta_A = \#u_j$'s in C_2 of P_A^p , and define γ_B and δ_B similarly for P_B^q . Since neither $c(r-1, s)$ nor $c(r, s-1)$ is in region 2, it follows that $C = C_1 + C_2 + c(r, s)$. By 2.12.2 of $P_A^p \sim P_B^q$, C contains the same number of t_i 's and u_j 's in P_A^p as in P_B^q . In both tableaux, let τ be the number of t_i 's, and μ be the number of u_j 's, in C . Since $c(r, s)$ contains t_i in P_A^p and u_j in P_B^q , it follows that:

$$\star \quad \tau = \alpha_A + \gamma_A + 1 = \alpha_B + \gamma_B, \quad \mu = \beta_A + \delta_A = \beta_B + \delta_B + 1.$$

Proof of 3.2.1: Suppose that $\alpha_B - \alpha_A = \beta_A - \beta_B = 1$. Then $\gamma_A - \gamma_B = \delta_B - \delta_A = 0$, so C_2 is either empty or contains an equal number of t_i 's and u_j 's in P_A^p as in P_B^q . Also, C_1 is nonempty, so $c(r, s+1)$ is in region 2 in P_A^p and in P_B^q . In P_B^q , $c(r, s)$ contains u_j , so by strict row inequality $c(r, s+1)$ contains t_i . The subsequent insertion steps are therefore

$$\begin{array}{c}
P_A^p \xrightarrow[\text{from } c(r, s)]{x \text{ bumps } t_i} P_A^{p+1} \xrightarrow[\text{row } r+1]{t_i \text{ enters}} \\
\\
P_B^q \xrightarrow[\text{from } c(r, s)]{x \text{ bumps } u_j} P_B^{q+1} \xrightarrow[\text{from } c(r, s+1)]{u_j \text{ bumps } t_i} P_B^{q+2} \xrightarrow[\text{row } r+1]{t_i \text{ enters}} .
\end{array}$$

Thus in both $P_A^p \rightarrow P_A^{p+1}$ and $P_B^q \rightarrow P_B^{q+1} \rightarrow P_B^{q+2}$, $c(r, s)$ is eliminated from region 2, and we are left with two separate components C_1 and C_2 (and with t_i to be inserted into row $r+1$). No change occurs in C_2 , so in both P_A^{p+1} and P_B^{q+2} , C_2 has $\gamma_A = \gamma_B$ t_i 's and $\delta_A = \delta_B$ u_j 's. Similarly, no change occurs in C_1 in $P_A^p \rightarrow P_A^{p+1}$, so C_1 of P_A^{p+1} contains α_A t_i 's and β_A u_j 's. On the other hand, in $P_B^q \rightarrow P_B^{q+1} \rightarrow P_B^{q+2}$, a single change occurs in C_1 , when the t_i in $c(r, s+1)$ is replaced with u_j . Thus C_1 of P_B^{q+2} contains $\alpha_B - 1$ t_i 's

and $\beta_B + 1$ u_j 's. But by 3.2.1, $\alpha_B - 1 = \alpha_A$ and $\beta_B + 1 = \beta_A$, so C_1 contains the same number of t_i 's and u_j 's in P_A^{p+1} as in P_B^{q+2} , and 2.12.2 is satisfied for $P_A^{p+1} \sim P_B^{q+2}$.

Now in both P_A^{p+1} and P_B^{q+2} , the only change that occurs in region 1 is that the same element x is added to $c(r, s)$, so 2.12.1 is satisfied. Similarly, as was already mentioned, both $P_A^{p+1} \rightarrow P_A^{p+2}$ and $P_B^{q+2} \rightarrow P_B^{q+3}$ consist of t_i entering row $r + 1$, so 2.12.3 is satisfied. It follows that $P_A^{p+1} \sim P_B^{q+2}$. This completes the proof of 3.2.1.

Proof of 3.2.2: The proof of 3.2.2 is dual, in a sense, to the proof of 3.2.1. Here are the details.

Suppose that $\alpha_B - \alpha_A = \beta_A - \beta_B = 0$. Then C_1 is either empty or contains an equal number of t_i 's and u_j 's in P_A^p as in P_B^q . Thus by (\star) , $\gamma_B - \gamma_A = \delta_A - \delta_B = 1$, so C_2 is nonempty, which implies that $c(r + 1, s)$ is in region 2 in P_A^p and in P_B^q . In P_A^p , $c(r, s)$ contains t_i , so by strict column inequality $c(r + 1, s)$ contains u_j . The subsequent insertion steps are therefore

$$\begin{array}{ccccccc}
 P_A^p & \xrightarrow{\substack{x \text{ bumps } t_i \\ \text{from } c(r, s)}} & P_A^{p+1} & \xrightarrow{\substack{t_i \text{ bumps } u_j \\ \text{from } c(r + 1, s)}} & P_A^{p+2} & \xrightarrow{\substack{u_j \text{ enters} \\ \text{column } s + 1}} & \\
 & & & & & & \\
 P_B^q & \xrightarrow{\substack{x \text{ bumps } u_j \\ \text{from } c(r, s)}} & P_B^{q+1} & \xrightarrow{\substack{u_j \text{ enters} \\ \text{column } s + 1}} & & &
 \end{array}$$

Thus in both $P_A^p \rightarrow P_A^{p+1} \rightarrow P_A^{p+2}$ and $P_B^q \rightarrow P_B^{q+1}$, $c(r, s)$ is eliminated from region 2, and we are left with two separate components C_1 and C_2 (and with u_j to be inserted into column $s + 1$). No change occurs in C_1 , so in both P_A^{p+2} and P_B^{q+1} , C_1 has $\alpha_A = \alpha_B$ t_i 's and $\beta_A = \beta_B$ u_j 's. Similarly, no change occurs in C_2 in $P_B^q \rightarrow P_B^{q+1}$, so C_2 of P_B^{q+1} contains γ_B t_i 's and δ_B u_j 's. On the other hand, in $P_A^p \rightarrow P_A^{p+1} \rightarrow P_A^{p+2}$, a single change occurs in C_2 , when the u_j in $c(r + 1, s)$ is replaced with t_i . Thus C_2 of P_A^{p+2} contains $\gamma_A + 1$ t_i 's and $\delta_A - 1$ u_j 's. But 3.2.2 and (\star) imply that $\gamma_A + 1 = \gamma_B$ and $\delta_A - 1 = \delta_B$, so C_2 contains the same number of t_i 's and u_j 's in P_A^{p+2} as in P_B^{q+1} , and 2.12.2 is satisfied for $P_A^{p+2} \sim P_B^{q+1}$.

Now in both P_A^{p+2} and P_B^{q+1} , the only change that occurs in region 1 is that the same element x is added to $c(r, s)$, so 2.12.1 is satisfied. Similarly, as was already mentioned, both $P_A^{p+2} \rightarrow P_A^{p+3}$ and $P_B^{q+1} \rightarrow P_B^{q+2}$ consist of u_j entering column $s + 1$, so 2.12.3 is satisfied. It follows that $P_A^{p+2} \sim P_B^{q+1}$. This completes the proof of 3.2.2. ■

3 The Proof of Theorem 5

Here we prove, for example, Theorem 5(b). The proofs of parts (c) and (d) of that theorem are similar.

Given $v \in a_{k,l}(n)$ and shuffle A , the (regular, dual)- A -RSK forms the tableau pair $(P^*, Q^*) = (P^*(v, A), Q^*(v, A))$ by applying the regular RSK to the t_i 's, and the dual conjugate RSK to the u_j 's of v under shuffle A . For simplicity, we refer to this algorithm as the dual- A -RSK. As in the A -RSK, P^* is the insertion tableau, and Q^* is the recording tableau of v under A . Here P^* is what we call a dual- A -SSYT; that is, it is weakly A -increasing in rows, and strictly A -increasing in columns.

Example 3.1. Let $k = 2$, $l = 1$, and $A : u_1 < u_2 < t_1 < t_2$. Let

$$v = \left(\begin{array}{c} 1 \cdots \cdots 4 \\ u_1, t_1, t_2, u_1 \end{array} \right).$$

Then,

$$v \xrightarrow{\text{dual-}A\text{-RSK}} \boxed{u_1} \quad \boxed{u_1 \mid t_1} \quad \boxed{u_1 \mid t_1 \mid t_2} \quad \begin{array}{|c|c|c|} \hline u_1 & u_1 & t_2 \\ \hline t_1 & & \\ \hline \end{array} = P^*,$$

and

$$Q^* = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array} \quad \blacksquare$$

Lemma 3.2. Let $v \in a_{k,l}(n)$, $A \in I$ and

$$v \xrightarrow{A\text{-RSK}} (P, Q), \quad v \xrightarrow{\text{dual-}A\text{-RSK}} (P^*, Q^*).$$

If v is non-repeating in its u -elements, then $P = P^*$ and $Q = Q^*$.

Proof. The A -RSK and the dual- A -RSK differ in only one rule: When some u_j enters a column under the A -RSK, it bumps the first element w_m such that $w_m > u_j$ (or if no such w_m exists, it settles at the end of the column). On the other hand, under the dual- A -RSK, u_j bumps the first element w_r such that $w_r \geq u_j$ (or settles at the end of the column). But u_j may appear only once in w , which implies that $w_r > u_j$, so this step is the same as that of the A -RSK. The proof now follows. \blacksquare

Notation. $v \in a_{k,l}(n)$ is said to be of type $(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l)$ if it is a permutation of $t_1^{\alpha_1} \cdots t_k^{\alpha_k} u_1^{\beta_1} \cdots u_l^{\beta_l}$.

Lemma 3.3. *Let $v \in a_{k,l}(n)$ be of type $(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l)$, and denote $\beta = \sum_{i=1}^l \beta_i$. Then there exists $w \in a_{k,\beta}(n)$ such that*

1. *The u -elements of w are non-repeating.*
2. *For every shuffle A , if $v \xrightarrow{\text{dual-}A\text{-RSK}} (P_v^*, Q_v^*)$, then there exists a corresponding shuffle A' of the elements of w such that $w \xrightarrow{\text{dual-}A'\text{-RSK}} (P_w^*, Q_w^*)$, where P_w^* is identical to P_v^* but with every v_i changed to w_i for all $i \leq n$. Consequently, $\text{sh}(P_v^*) = \text{sh}(P_w^*)$.*

Proof. To avoid confusion between the elements of v and of w , we let u'_1, \dots, u'_l denote the u -elements of v .

Form the sequence w from v as follows. Replace the u'_1 's in v with u_1, \dots, u_{β_1} , moving from right to left. Replace the u'_2 's with $u_{\beta_1+1}, \dots, u_{\beta_1+\beta_2}$, moving from right to left. Continue in this way until u'_l , and including the u'_l 's.

Clearly the u -elements of w are non-repeating, satisfying 3.3.1.

Given some shuffle A of the elements of v , define the shuffle A' of the elements of w as follows. For every $i \in \{1, \dots, k\}$,

$$\begin{aligned} t_i <_A u'_1 &\implies t_i <_{A'} u_1 <_{A'} \dots <_{A'} u_{\beta_1} \\ t_i <_A u'_2 &\implies t_i <_{A'} u_{\beta_1+1} <_{A'} \dots <_{A'} u_{\beta_1+\beta_2} \\ &\vdots \\ t_i <_A u'_l &\implies t_i <_{A'} u_{\beta_1+\dots+\beta_{l-1}+1} <_{A'} \dots <_{A'} u_{\beta}, \end{aligned}$$

$$\begin{aligned} u'_1 <_A t_i &\implies u_1 <_{A'} \dots <_{A'} u_{\beta_1} <_{A'} t_i \\ &\vdots \\ u'_l <_A t_i &\implies u_{\beta_1+\dots+\beta_{l-1}+1} <_{A'} \dots <_{A'} u_{\beta} <_{A'} t_i. \end{aligned}$$

We compare the A -RSK insertion of the v 's with the A' -RSK insertion of the w 's. Note that the shuffle A and its derived shuffle A' are similar in that $v_i <_A v_j \implies w_i <_{A'} w_j$, but they differ in one fundamental way: For $i < j$ such that v_i, v_j, w_i and w_j are u -elements, $v_i =_A v_j \implies w_i >_{A'} w_j$. Now, if w_j reaches a cell inhabited by $w_i >_{A'} w_j$, then it bumps w_j to the next column, just as v_j would bump $v_i =_A v_j$ to the next column under the dual- A -RSK. On the other hand, if w_i reaches a cell inhabited by $w_j <_{A'} w_i$, it settles below w_j , whereas v_i would bump $v_j =_A v_i$ to the next column. However, such a situation never occurs, since $i < j$ and $v_i =_A v_j$ implies that every column reached by w_j is first reached by w_i . The proof of this is as follows.

Suppose that for some x , $w_i = u_{x+1}$ and $w_j = u_x$. Then every column reached by w_j is first reached by w_i , by induction on the columns of P_w^* . Trivially, w_i reaches column 1 before w_j . By the induction assumption, $w_i = u_{x+1}$ is in column c' , $c' \geq c$. If $c' > c$, then we are done. Assume $c' = c$: $w_i = u_{x+1}$ is already in

column c , and $w_j = u_x$ is inserted into column c . It bumps the first w_d such that $w_d \geq w_j = u_x$. Now $v_i =_A v_j$ implies that there does not exist any t_z such that $w_j <_{A'} t_z <_{A'} w_i$. Hence $w_d = u_{x+1} = w_i$ is bumped to column $c + 1$.

This clearly extends to the general case $i < j$, $v_i = v_j$, $w_i = u_y$, $w_j = u_x$, for general $y > x$.

Hence the steps of the dual- A' -RSK on w are identical to the steps of the dual- A -RSK on v , but with every v_i , $i \leq n$, changed to w_i . This implies that 3.3.2 is satisfied for w . ■

Example 3.4. Let $v = t_2 u_2 u_1 u_1 t_1$, and $A = t_1 < t_2 < u_1 < u_2$. The sequence $w = t_2 u'_3 u'_2 u'_1 t_1$ clearly satisfies 3.3.1; we show that it satisfies 3.3.2 for shuffle A , by letting $A' = t_1 < t_2 < u'_1 < u'_2 < u'_3$. Under shuffles A and A' ,

$$v \xrightarrow{A\text{-RSK}} (P_v^*, Q_v^*) \quad \text{and} \quad w \xrightarrow{\text{dual-}A'\text{-RSK}} (P_w^*, Q_w^*),$$

where

$$P_v^* = \begin{array}{|c|c|c|c|} \hline t_1 & u_1 & u_1 & u_2 \\ \hline t_2 & & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline v_5 & v_4 & v_3 & v_2 \\ \hline v_1 & & & \\ \hline \end{array}$$

$$P_w^* = \begin{array}{|c|c|c|c|} \hline t_1 & u'_1 & u'_2 & u'_3 \\ \hline t_2 & & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline w_5 & w_4 & w_3 & w_2 \\ \hline w_1 & & & \\ \hline \end{array}$$

Thus 3.3.2 is satisfied for shuffle A . ■

We can now give

The Proof of Theorem 5(b). Let v be of type $(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l)$, and denote $\beta = \sum_{i=1}^l \beta_i$. Lemma 3.3 implies that there exists a sequence $w \in a_{k,\beta}(n)$ with no repeating u -elements, and with shuffles A', B' such that

$$w \xrightarrow{\text{dual-}A'\text{-RSK}} (P_{A'}^*, Q_{A'}^*), \quad w \xrightarrow{\text{dual-}B'\text{-RSK}} (P_{B'}^*, Q_{B'}^*),$$

where $\text{sh}(P_{A'}^*) = \text{sh}(P_A^*)$ and $\text{sh}(P_{B'}^*) = \text{sh}(P_B^*)$. Since w contains no repetitions in its u -elements, Lemma 3.3 implies that

$$w \xrightarrow{A'\text{-RSK}} (P_{A'}^*, Q_{A'}^*), \quad w \xrightarrow{B'\text{-RSK}} (P_{B'}^*, Q_{B'}^*).$$

Thus by Theorem 2, $\text{sh}(P_{A'}^*) = \text{sh}(P_{B'}^*)$, which implies our result. ■

The proofs of parts (c) and (d) of Theorem 5 are similar to that of Theorem 5(b), since Lemma 3.3 can be applied also to the (dual, regular)- A -RSK and the (dual, dual)- A -RSK. Both algorithms are t -dual; for simplicity, let t'_1, \dots, t'_k denote the t -elements of v . The t 's of the sequence w of Lemma 3.3 for parts (c) and (d)

are set as follows. Replace the t'_1 's in v with t_1, \dots, t_{α_1} , moving from left to right. Replace the t'_2 's with $t_{\alpha_1+1}, \dots, t_{\alpha_1+\alpha_2}$, moving from left to right. Continue in this way until t'_k , and including t'_k .

Since the (dual, regular)- A -RSK of part (c) is u -regular, the u 's of w are identical to those of v . However, the (dual, dual)- A -RSK of part (d) is u -dual, so in this case the u 's of w are derived the same way as in the proof of Lemma 3.3. Finally, shuffle A' is derived from A in parts (c) and (d) by methods analogous to that of part (b).

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